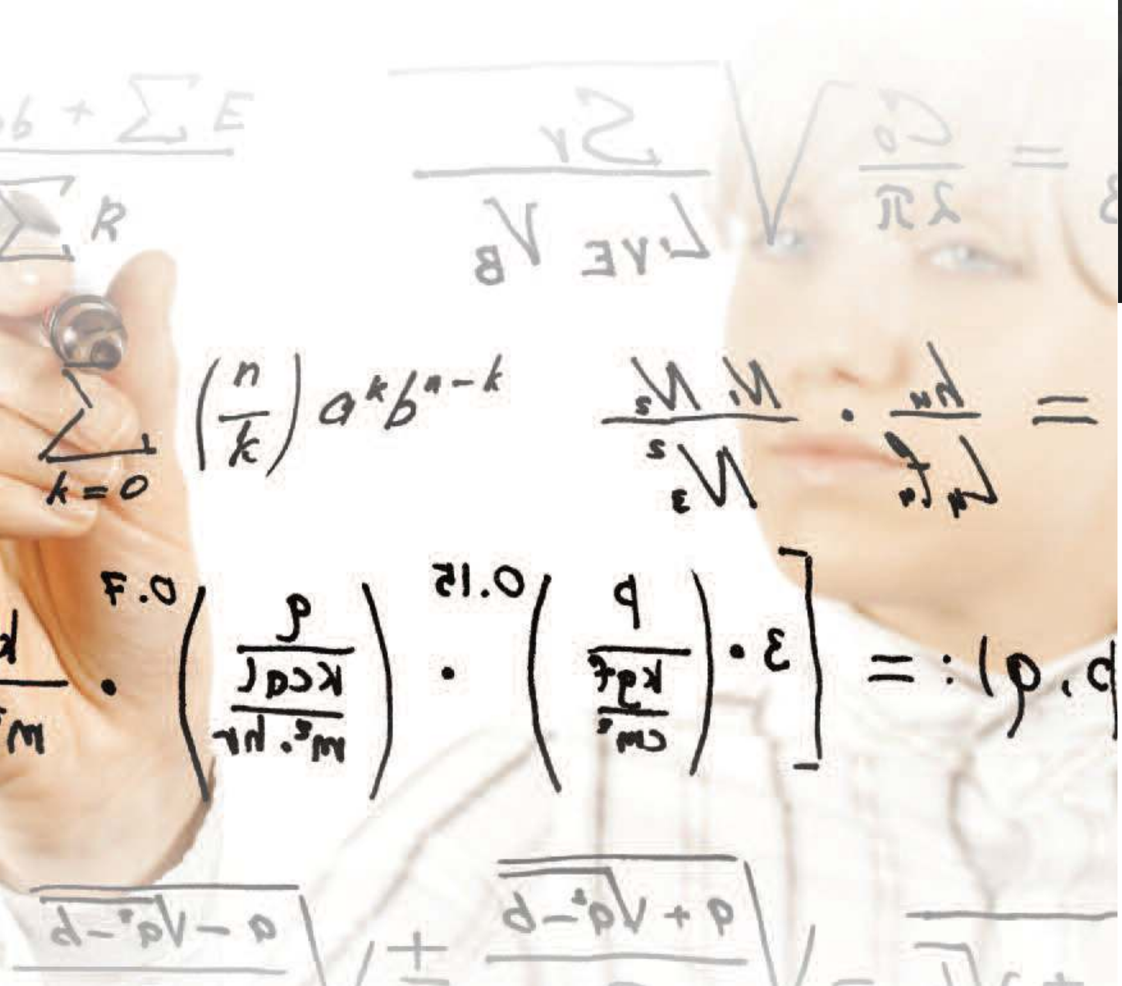
Where the Maths you learn is used

Queen Mary
University of London



Queen Mary
University of London

Thanks to the HE STEM Programme, the Institute for Mathematics and its Applications, and the Queen Mary Student Experience Investment Fund for funding work towards this project.

This booklet was compiled by Vivien Easson and Michael Yates with the help of the following staff and research students of the School of Mathematical Sciences, Queen Mary: Andy Drizen, Joe Elliston, John Faben, Jonit Fischmann, Nicholas Illman, Saiful Islam, Mudakkar Khadim, Jennifer Klim, Javier López Peña, Ben Parker, Wei Roberts, Manjula Samarasinghe, Lynne Stockman, Emil Vaughan and Francis Wright. Thanks also to David Arrowsmith, Peter McOwan, Abi Sharma and Hugo Touchette for advice.

maths careers .ORG.UK

Institute of mathematics & its applications

National HE STEM Programme

Contents

Introduction (p3)

Business and Management (p6-11)
- Lognormal distribution
- Polynomials and modelling
- Hypothesis testing
- Long tail distributions and sales
- Arbitrage

Finance and Economics (p12-17)
- Differential equations and modelling
- The Black-Scholes formula
- Pricing derivatives and options
- Monte-Carlo algorithms
- Utility functions

Sound and Graphics (p18-23)
- Mathematics and music composition
- Sound waves
- Vibration and harmonics
- Tea for two… or four… or eight?
- Fractals and virtual worlds

Space and Navigation (p24-29)
- Aerodynamics
- Gravity
- Relavity and GPS
- Networks and logistics
- Polar coordinates

Communications and Security (p30-33)
- Simple encryption
- Internet security and digital signatures
- Error-correcting codes
- Data compression and digital music
- Connection speeds and processing power

Energy and the Environment (p34-37)
- Wind and wave power
- Extreme weather events
- Seasonal variability
- Order and chaos
- Modelling the natural world

To Infinity and Beyond (p38)

Explicit links from the mathematics curriculum to applications in science, technology, business and industry....

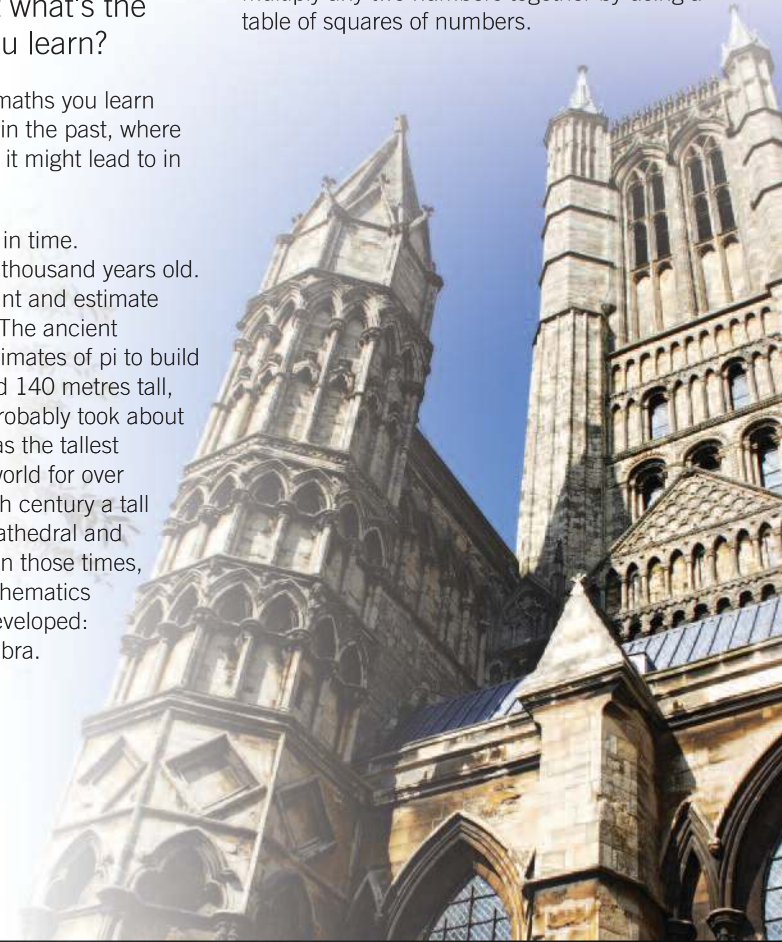
Introduction

Any university degree will help you develop your thinking skills and improve your chances of getting a job when you leave. A degree in mathematical sciences will certainly do both of those. But what's the point of the maths you learn?

Here we explain where the maths you learn at university has been used in the past, where people use it now, and what it might lead to in the future.

So first let's take a trip back in time. Mathematics is at least four thousand years old. Of course, people could count and estimate distances long before then. The ancient Egyptians used accurate estimates of pi to build gigantic pyramids. At around 140 metres tall, the Great Pyramid of Giza probably took about twenty years to build and was the tallest manmade structure in the world for over 3800 years. In the fourteenth century a tall spire was built on Lincoln Cathedral and this took the record. Between those times, most of the compulsory mathematics now taught in school was developed: geometry, trigonometry, algebra.

Mathematics was created all around the world. From the Maya of South America to Greece, India and Arabia, mathematics was useful. The Babylonians living four thousand years ago in what is now Iraq used an algebraic formula involving the difference of two squares to multiply any two numbers together by using a table of squares of numbers.



Introduction cont.

Around the year 600, mathematics was taught as part of the civil service examinations in China, to aid with practical problems such as surveying land, civil engineering and distribution of resources. Chinese textbooks from this period include university-level methods such as Gaussian elimination to solve systems of simultaneous linear equations, and many other algebraic and geometric techniques.

In the computer game Civilisation, you start out with a small society and develop it through the ages, learning technologies as you go. Mathematics is needed in order to build bigger buildings and structures, more powerful

weapons and to organize your society as it grows. Just as in the game, so it was throughout history. Many great mathematicians have worked as advisors to governments or leaders. Archimedes used his own advances in geometry and mechanics to invent war engines used in Sicily against the Roman army in 212 BCE. Joseph Fourier was Napoleon's scientific advisor when the French invaded Egypt in 1798. He later produced a famous memoir on the mathematical theory of heat diffusion.

Statisticians and mathematicians continue to work for governments around the world. In the UK, the Office for National Statistics compiles and publishes a wide range of economic and social statistics which help the government and businesses plan for the future. The Government Actuary's Department provides consultancy on pensions and insurance. Mathematicians work in the Government Communication Headquarters, one of the UK's three intelligence agencies. There are many other jobs in the public and private sector which require mathematical skills, and you can see some of them in this booklet or on **www.mathscareers.org.uk**.

Whether you enjoy working in a team or on your own, work for a small company or a large multinational, a mathematics degree will help you get where you want to be. So read on!

“There are many other jobs in the public and private sector which require mathematical skills, and you can see some of them in this booklet.”

"Accountants obviously need to be good with figures. They also need to be able to communicate what they find to managers or shareholders"

Business and Management

Accountancy is the first obvious link between the world of business and mathematics. Almost all businesses need to have their accounts audited by external firms of accountants or by their own in-house finance team. Accountants obviously need to be good with figures. They also need to be able to communicate what they find to managers or shareholders. In all areas of business – and even to get an interview for a job! – you need to be able to write clearly.

There is a clear link between proficiency in mathematics and a career in accountancy but there are less well known applications of topics in the mathematics curriculum to activities carried out by businesses. Economic and business modelling are vital for companies to gain insight about what they should sell or how they should plan for the future. Actuaries need to work out whether a country or a company will have enough money in the future to pay pensions to its retired citizens or staff. To do this they have to know how long people are likely to live, which means looking at trends in healthcare and population. Making sound predictions in an unsure world is increasingly important, which is why a good understanding of probability and statistics is highly valued by many employers.

Here are some examples of topics taught in university mathematics being used in business and management.

Business and Management cont.

Lognormal distribution

The normal distribution is a well-known distribution which when plotted on a graph is bell-shaped and symmetrical. Data sets such as “the heights of adult men in London” follow a normal distribution: there are many men of medium height but smaller numbers of very tall or very short men.

The lognormal distribution is derived from the normal distribution: a random variable X has a lognormal distribution if log(X) has a normal distribution. However, it has very different properties. This distribution is skewed to the right and hence more extreme values are more likely to occur. It turns out that it is this property that makes it a very good distribution to model financial processes. For example, the modelling of price changes in shares is lognormally distributed. The “skewness” of the distribution is used to model insurance losses like natural catastrophe damage since this distribution allows for the probability of a very high loss not to be underestimated.

Polynomials and modelling

Polynomials are simple algebraic expressions that contain both constants and variables, which generate a curve when plotted on a graph. Designers of roller coasters might use them to design the curves on their rides. In business they are used to model stock markets to view how prices will change over time, indeed combinations of polynomial functions can be used for this purpose. The advantage of using polynomials is that you can change a given variable and then observe the overall effect. For example, in business, they can be used to look at the effect on sales should you wish to raise the price of an item you’re selling. However to understand how well a polynomial function really models your data, you need to have a good understanding of the principles of statistical modelling. Someone who’s studied statistics at university should be able to say how accurate their estimates or models are!

Hypothesis testing

Hypothesis testing is a good way of comparing two things against each other to see an effect or change from one to the other. For example in the workplace you might want to decide whether a training course has actually improved staff performance. A hypothesis you might use would be phrased as a statement “The quality of people’s work is no better after training than it is before”. The experiment that you conduct will seek to disprove this statement. If you can disprove the statement that means that you have some good evidence to show that the training course was helpful.

Suppose that the training course is in using a new financial records system, and that you measure how long it takes five members of staff to process ten invoices. Before the training you record the number of minutes it takes each person to carry out this task and get scores of 61, 80, 45, 61 and 72 minutes. After the training you record the times taken for the same type of task and get scores of 54, 62, 42, 56 and 60 minutes. These times look faster. Is that just chance or did the training really help?

Sources of variation might be staff motivation or how well the IT systems were working at the time. Your analysis therefore needs to take account of these variations to produce results that are of “statistically significant” difference. This then gives an accurate picture of what the effect was of the training provided, so that a manager can decide whether it’s worth paying for in the future. Consultancy firms use this technique within their business management strategies when employed by clients to improve processes or outcomes within the company.

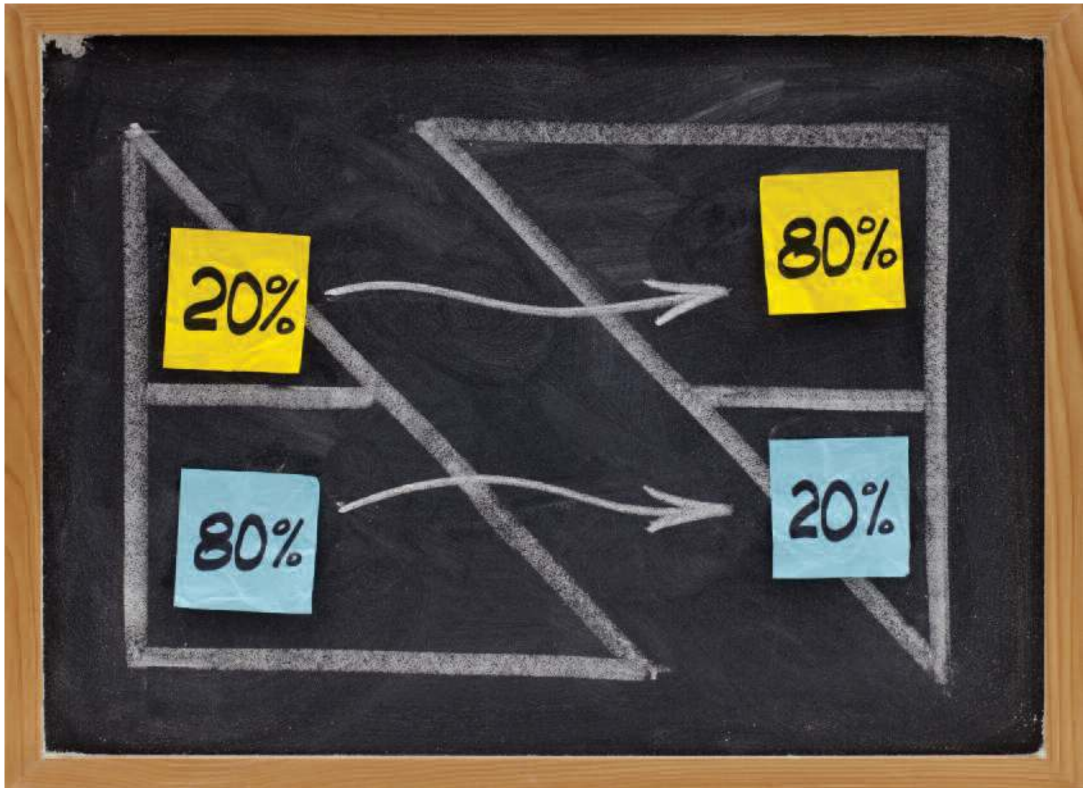
Business and Management cont.

Long tail distributions and sales

A common question asked in business is whether a particular company should focus on selling extremely popular items which are sold by many other businesses, or focus on selling less popular items which are harder for a customer to find. For example, the book section in a large supermarket might only stock bestsellers, whereas a small military history bookstore might specialise in rare military history books and avoid selling bestsellers.

You might think that selling popular items would be the best way to go since more customers are likely to want to buy them. However there are advantages of the other approach, particularly for an internet business where it's easier and cheaper to store less popular items for a long time until the perfect buyer comes along. Statistical modelling of the population of customers shows that there are more people in the population in the tail of this kind of probability distribution than there would be in a normal distribution. This long tail represents the fact that a lot of people are willing to buy items which are generally less popular but which suit their own tastes and preferences. This kind of distribution is known as a power-law distribution or Pareto distribution.

Probability and calculus modules can help you understand the mathematics behind these differences. For example, learning about different distributions and profit models teaches you the underlying theory for exploring the relationship between how well products sell and how many it would be best to stock to make the best profits. Statistical models can be applied to business models such as portfolio optimization, capital asset pricing or efficient market hypotheses.



Arbitrage

The idea of arbitrage is fundamental to the understanding of financial transactions. Arbitrage is the practice of taking advantage of a price difference between two markets: in theory it leads to a risk-free profit. The arbitrage theorem can be used to minimize the risk of a portfolio as follows. Suppose that you have a fixed amount of money which you would like to invest by buying a number of different shares.

The arbitrage theorem can help you determine the best way of investing. Many hedge funds are searching the financial markets for arbitrage opportunities: remember this means making money without risk. In order to protect currencies, exploiting arbitrage opportunities in currency markets is forbidden.

Finance and Economics

Financial mathematics is an area of study that has grown rapidly over the past fifty years. It draws on many topics in mathematical sciences: calculus, analysis, partial differential equations, numerical methods and probability. Many financial models rely on stochastic differential equations, which are differential equations where one of the terms is a stochastic, or random, process. The idea is that we might know what the current situation is, but there is uncertainty about how things will develop in the future. For example, it's difficult to predict whether the price of shares in a certain company will go up or down. However we may be able to say that it's very unlikely that the price will suddenly rise or fall by a large amount. Understanding how to model this uncertainty is one of the key ideas in financial mathematics.

Differential equations and modelling

Differential equations can be used to assess the rate of change of some quantity over time. There are obvious uses for this in physics. For example it's straightforward to work out braking distances for a car using differential equations. But in a real accident the distance it takes for a car to stop will be longer than the theoretical distance because of the time needed for the driver to see and react to a hazard. Any mathematical model needs to take this into account.

Finance and Economics cont.

In finance, it's often important to know what the value is now of knowing that you will receive some amount of currency at a particular future time. For example, how much would you pay now in order to be given £1000 in twelve months time? Alternatively, how much should someone pay you now in order for you to sign a contract saying that you would give them £10,000 in three years' time? These questions are linked to interest rates. There are many mathematical tools which can help analyse these questions, including matrices, polynomial approximation and linear programming.

This can also be applied to the field of economics. A nation's economic growth is the rate of change of a country's GDP, for example the UK economy may grow 2% per annum. Figures such as these can be used to build equations which describe economic growth. However there will often be factors which are difficult to model: perhaps the equations aren't easy to solve, or there may be too much uncertainty. It's important to remember that what works in theory may not work in practice!

The Black-Scholes formula

The Black-Scholes formula is used for pricing complicated financial products that depend on changes of the stock market. It's part of a mathematical model that was first described by Fischer Black and Myron Scholes in 1973, and then expanded by Robert Merton. This discovery won Scholes and Merton the Nobel Prize in Economics – it could not be awarded to Black as he had died two years before the prize was awarded.

The ability to be able to price financial derivatives was crucial for the extensive development of investment banking. Recently the model has been subject to criticism since the financial crisis of 2008 was partly due to faulty pricing. However any other model would most likely involve similar techniques using stochastic differential equations.

Pricing derivatives and options

In finance, a derivative is a contract between two parties which has a value determined by the price of something else. An example would be a contract entitling me to buy 100 shares of a particular company at £20 each in a month's time. If I think the price of these shares is going to rise over the next month, I'll be willing to pay more to sign this contract than if I think their price will fall. This is called an option – I have the option to buy these shares at this price, but if I don't want to buy them I don't have to. There are much more complicated examples.

The mathematical formulae that were derived to describe the price of financial options are constantly used in the stock market. There are two reasons for buying options. Initially they were introduced as a risk management tool. Someone bought a share or a commodity and wanted to make sure they were able to sell it at a later point for a given price. Nowadays options are mainly a tool for speculative trading as they allow for greater profits than simple stock trading.

The price of an option is not easy to give as it depends on the following: the price of the underlying commodity or share; the price the buyer wanted to secure; the sell time and also the current risk free interest rate. Different methods are used to price derivatives and options. As well as the Black-Scholes model and variations on it, Monte Carlo methods may be used to simulate what might happen and figure out a price for the option by averaging over lots of different possibilities.

Finance and Economics cont.

Monte Carlo algorithms

Monte Carlo is a district of Monaco popularly known for its casino. Monte Carlo methods are computational algorithms based on repeated random sampling to compute the results. Their applications can be found in almost all areas of science and have been used to develop strong game-playing computer programs. They are typically used when it's difficult to compute what might happen exactly by mathematical analysis.

The idea is as follows. First you need to define what inputs you will allow. Next you need to select randomly from these inputs using a given probability distribution. Then carry out a sequence of fixed operations on the chosen input to get an output. Finally, combine all of the outputs from many inputs to get an overall picture of the kinds of outputs that arise.
For example, the inputs might be all possible sequences of the next ten moves in a game and the operation would be a calculation evaluating how strong your position is at the end of one of those sequences of ten moves. For each of your initial moves you can average the results of the calculations for all the sequences starting with it. A move with a good average score is more likely to be a strong move.

Note that Monte Carlo algorithms don't look at all of the possible inputs but only a random sample of them. You might get unlucky: you might miss a brilliant move because you never happened to pick a sequence to look at which started with that move. In the long run, however, you'll be successful.

Utility functions

In economics, a utility function is a way of expressing mathematically a consumer's preference for one consumable over another. For example, scoring books on a scale of zero to ten gives a very simple utility function! By scoring one book higher than another I'm expressing a preference for that book over the other.

More complicated utility functions involve ideas of probability. Which would you prefer of the following: being given £10, or tossing a coin and getting £21 if it comes up heads but nothing if it comes up tails? What if the sums involved were £1 million and £2.1 million? What if the money was for your best friend and not for you?

"Which would you prefer of the following: being given 10 pounds, or tossing a coin and getting 21 pounds if it comes up heads but nothing if it comes up tails?"

Sound and Graphics

Mathematics has influenced the development of art and music for thousands of years. Perspective or technical drawing requires careful geometrical calculations of angles and lengths. Musical notes which sound good together have frequencies whose ratios are simple fractions such as 4:3 or 5:4. Fractal images are used to provide landscapes in virtual game worlds, and mathematical sequences inspire new genres of music.

Mathematics and music composition

Modular arithmetic is a key topic in both mathematics and music. It's the kind of arithmetic we carry out whenever we calculate at 11am that it's three hours until 2pm. On a clock face, 11+3=2. This is arithmetic modulo 12: we take the result 11+3=14 and subtract multiples of 12 to get an answer between 0 and 12.

Sound and Graphics cont.

Modular arithmetic is vital in mathematics because it is one of the foundations of abstract algebra. Number theory, group theory and ring theory are all part of abstract algebra – this is mathematics which looks at a system and extracts its most important symmetries, properties and rules. Modular arithmetic is used every day all around the world: internet security encryption depends on it.

It also has applications in music. Arithmetic modulo 12 is used in the twelve tone equal temperament system. Equal temperament is a system of tuning where adjacent notes have the same frequency ratio. This system is the one most commonly used in classical music. The twelve notes in the musical octave are called C, C#, D, D#, E, F, F#, G, G#, A, A#, B and then it loops around back to C again. Many of the great classical composers such as Bach or Beethoven used mathematical tricks involving this structure to devise complex symphonies that are still highly popular today.

As well as modular arithmetic, the modern Greek composer Iannis Xenakis used probability theory and group theory as well as ideas from architecture and art to create his compositions. The 2001 Tool single Lateralus uses the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21,… in its lyrics and in the different rhythms and time signatures.

Sound waves

Musical notes and other sounds can be represented by sound waves. A pure tone can be represented by a sine wave with a given frequency. For example, middle A is the note that an orchestra uses to tune up and it has a frequency of 440Hz, which means 440 peaks and troughs per second. Other sounds can be represented by more complicated waves.

A Fourier series is a sum of sine and cosine waves. It can be an infinite sum of terms of the form c sin(nx) where the amplitude coefficient c gets smaller as the frequency coefficient n gets larger. If the coefficients get small enough as n increases, the series will converge.

Any continuous unchanging sound can be represented by a periodic function and then expressed as a Fourier series. If the periodic function being decomposed into Fourier components is odd then only sine terms will contribute. If the function is even then only cosine terms will contribute. Remember that a function f is odd if f(x) = -f(-x) and even if f(x)=f(-x).

Vibration and harmonics

A harmonic of a sound wave is a wave whose frequency is an integer multiple of the frequency of the original sound wave. Niccolo Paganini was an Italian violinist who was born in Genoa in 1782. Some people thought that Paganini was possessed by the devil. His appearance dressed all in black was partly to blame. However it was his use of harmonics that really contributed to the myth. Usually as a violinist's hand moves towards their shoulder then the notes rise in frequency and therefore sound higher in pitch. With harmonics, however, the process is the opposite. Those who did not realise this fact could have seen this apparent break with the laws of nature as further evidence that something devilish was going on!

Sound and Graphics cont.

Tea for two or four or eight?

A nice experiment that can be reproduced in a lecture is the teacup experiment. For a large lecture this requires a video camera linked to a projector and a microphone. The lecturer places a teacup on the front bench and hits the rim with a teaspoon. If the rim is struck above the handle, opposite the handle or at periodic points exactly a quarter of the circumference away then the frequency is low (four points in total).

If the cup is struck at points halfway between those mentioned above then the frequency is higher. It is the presence of the handle that restricts oscillations about that point. To understand this, note that a node in a wave is a point where the amplitude of the sound wave is at a minimum: for example when x=0 in sin(x). An anti-node is a point where the amplitude of the sound wave is largest: for example when x=0 in cos(x).

Striking the cup on the handle will set up an anti-node. However the handle will oppose this, resulting in a gain in frequency. Striking the other points does not suffer this effect since the handle is then a node of the wave.

Fractals and virtual worlds

Computer graphics are far more advanced now than in the days of ASCII art and 640 x 480 resolution. Massively multiplayer online games and real-time strategy games require complex virtual worlds and impressive graphics. It would be far too time-consuming to program the graphics for each tree or plant separately.

Instead, games programmers rely on mathematics to make their lives easier. A fractal is an object where magnifying a part of it gives you another object which looks fairly similar to the bit you started with. For example, a twig of a tree looks fairly similar to a branch, it's just smaller. There are many objects in the natural world which have fractal properties: mountains, lightning bolts, clouds, snow flakes, coastlines… enabling a realistic-looking world to be created much faster.

Space and Navigation

> "Whether it's navigating the galaxy, the surface of the Earth or your local supermarket, maths can help"

Mathematics is full of tools which help us describe positions and journeys in space. Whether it's navigating the galaxy, the surface of the Earth or your local supermarket, maths can help. Engineers also use mathematical techniques to help design cars, ships, planes and spacecraft.

Aerodynamics

Imagine that you're beginning to test a new design for an aeroplane by looking at a simulation of the plane in a wind tunnel. The value of the Reynolds number will be the key thing you look at here. Low values of the Reynolds number mean that the flow is smooth and constant. High values are bad news: they mean that the flow is turbulent and has lots of vortices and instabilities. The Reynolds number has been used in fluid dynamics for over a century and so is a well-tested concept. That's a good thing if it's going to be used to keep aeroplanes in flight!

Space and Navigation cont.

Fighter pilots feel large g-forces when they make quick changes in direction. In preparation for this they sit in a piece of equipment called a centrifuge which spins them round and round in different directions. They get used to coping with this artificially induced gravity from the centrifuge before they have to go out and do it for real in their fighter jet. This is an example of a central force as discussed in mathematical dynamics.

Gravity

Gravity on a bigger scale is even more complicated. Across our Solar system, experiments can detect the influence of gravity curving the fabric of spacetime. The maths required for this starts with elementary geometry, differential equations and vector spaces and then combines them to produce something called differential geometry. Understanding differential geometry for surfaces is the first step towards understanding higher dimensional spaces.

One application of differential geometry is the description of gravity given by Einstein's General Theory of Relativity via the curvature of space-time. Given two different frames of reference in the theory of relativity, the transformation of the distances and times from one to the other is given by a linear mapping between two vector spaces.

Relativity and GPS

The Global Positioning System of satellites in orbit around the Earth is well known and used by millions of people, particularly with car satellite navigation. The satellite beams information about its position and the time on the atomic clock on board. A GPS receiver picks up a number of these and uses this information to pinpoint its location. The method used is extremely sensitive to the time measurements. It's therefore crucial to get the atomic clocks on the satellites absolutely accurate.

Relativity causes two problems here. Special relativity tells us that the faster an object moves relative to us, the slower its onboard clock runs. With orbital speeds around four kilometres per second this adds up to an error of around seven microseconds per day. General relativity has a larger effect. Because the satellites are further from the Earth's gravitational potential than we are, the satellite clocks will run faster than ours due to the gravitational frequency shift. This effect is about 46 microseconds per day.

Combining both the above effects we get the result that the satellite clocks run faster than ours by about 39 microseconds per day. This may seem really small. However, this error would give a discrepancy of 10km per day and worse, this effect would grow cumulatively! To account for relativistic effects, the clocks on the satellites are set to a slightly off-set frequency which is calculated to exactly counteract the relativistic effects. Specifically, this is 10.22999999543 MHz instead of 10.23 MHz!

Space and Navigation cont.

Networks and logistics

It is a strange fact of the UK's rail network that, when travelling from London to Manchester, it is sometimes cheaper to buy a series of tickets (London to Derby, Derby to Manchester) than it is to buy a direct ticket. To find the cheapest fare, we could make an abstract graph of points (vertices) and lines (edges) joining them. The vertices would be UK railway stations. We could join each pair of stations by an edge which is "weighted" by how much it costs to travel on that edge. Then we could use the shortest path algorithm to find the cheapest route between any pair of stations. I don't know of any third-party website which has implemented this but I'd certainly use it if one existed!

Here's another scenario. A refuse collection team have a number of streets for which they are responsible. They must drive down all the streets and empty the rubbish bins left outside. The council buy the diesel for the lorries and pay the wages of the staff. Obviously they want to keep their costs as low as possible.

One of the ways of lowering costs is choosing the fastest route for the lorry to take. For example, suppose the lorry collects garbage from one street, then returns to the depot, then moves off to another street, and returns to the depot and continues until all the streets are covered. This is probably not optimal unless your streets are laid out in a star formation with the depot in the centre – quite unlikely! An algorithm called the Chinese Postman algorithm will provide you with a cheapest route to solve this problem. The algorithm is called this because it was discovered by the Chinese mathematician Kwan Mei-Ko who wrote about it in an article published in 1962.

Polar coordinates

Any system with radial force is an ideal candidate for using polar coordinates. In navigation pilots use polar coordinates because it is more efficient to give an angle to specify their direction from a given object.

Calculations using complex numbers in the Cartesian form are frustrating. Each number $x+iy$ when multiplied or divided by another complex number, generates lots of different terms. Polar coordinates make things much easier. The phase parts only need to be added or subtracted, unlike the multiplications and additions of real and imaginary parts in the Cartesian case. This makes multiplication or division quick and trivial.

Communication and Security

With more and more of our daily tasks being done online – from paying the electricity bill to chatting with friends – keeping internet transactions safe and secure is ever more vital. We need to be able to rely on the information we send being received accurately at the end. We need to know that our credit card details aren't being intercepted by someone else when we buy goods online.

Simple encryption

In the early days of internet forums, answers to puzzles were hidden by a simple substitution cipher called ROT13. This stands for "rotate by 13" and involves replacing each letter by the letter thirteen places further on or back in the alphabet. It's very easy to break and should not be used anywhere where secrecy is important.

While not widely used nowadays, modern bit-oriented block ciphers can be viewed as substitution ciphers on large binary alphabets. The main example of this is the Data Encryption Standard (DES) and its successor the Advanced Encryption Standard (AES). These are encryption standards used by the US government in protecting data.

"We need to be able to rely on the information we send being received accurately at the end"

Internet Security and digital Signatures

Security of the internet is vitally important in today's world, especially as activities on it become more popular and it becomes more accessable. RSA is the main algorithm used to do this and its use of sufficiently long keys ensures that it is secure. It is widely used in electronic commerce. It relies on a few simple theorems in number theory and the fact that factorizing large numbers takes a very long time.

It's easy to work out what 5711x4229 is but much harder to find the factors of 37448951.

Authentication on sites on the internet is important to prevent identity theft. It also helps prevent illegal programs, viruses and trojans from infecting your computers. Digital signatures in digital certificates verify identities of organizations and are used on websites when dealing with confidential data.

Communication and Security cont.

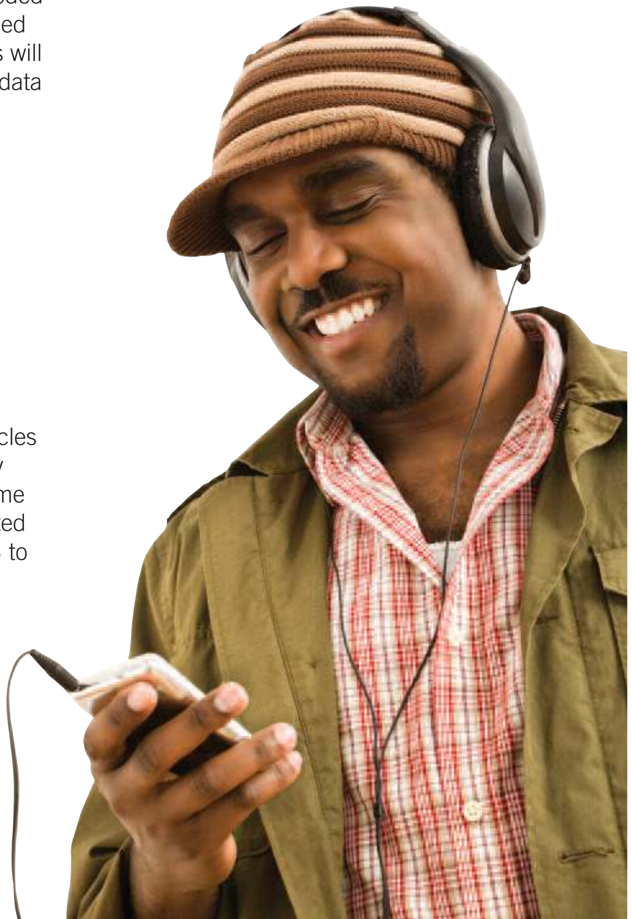
Error-correcting codes

Hamming codes are simple error-correcting codes. Your computer's memory stores information as Hamming codes, which allows it to detect and correct errors in the data. The Hamming code encodes data as words called codewords, which are then stored and decoded when needed. These codewords are designed so that a small number of changes or errors will not produce another valid codeword. If the data gets partially corrupted the computer can choose the codeword which is closest, correcting the errors.

Data compression and digital music

Fractals have been used as tools for compressing large amounts of data. For example, Microsoft released Encarta Encyclopedia on CD and used fractal data compression to compress thousands of articles into 600 megabytes of data. This is done by taking parts of the data and storing it by some function. The Mandelbrot set is a complicated fractal. Its image would normally take 35KB to store but takes up only 7 bytes after compression!

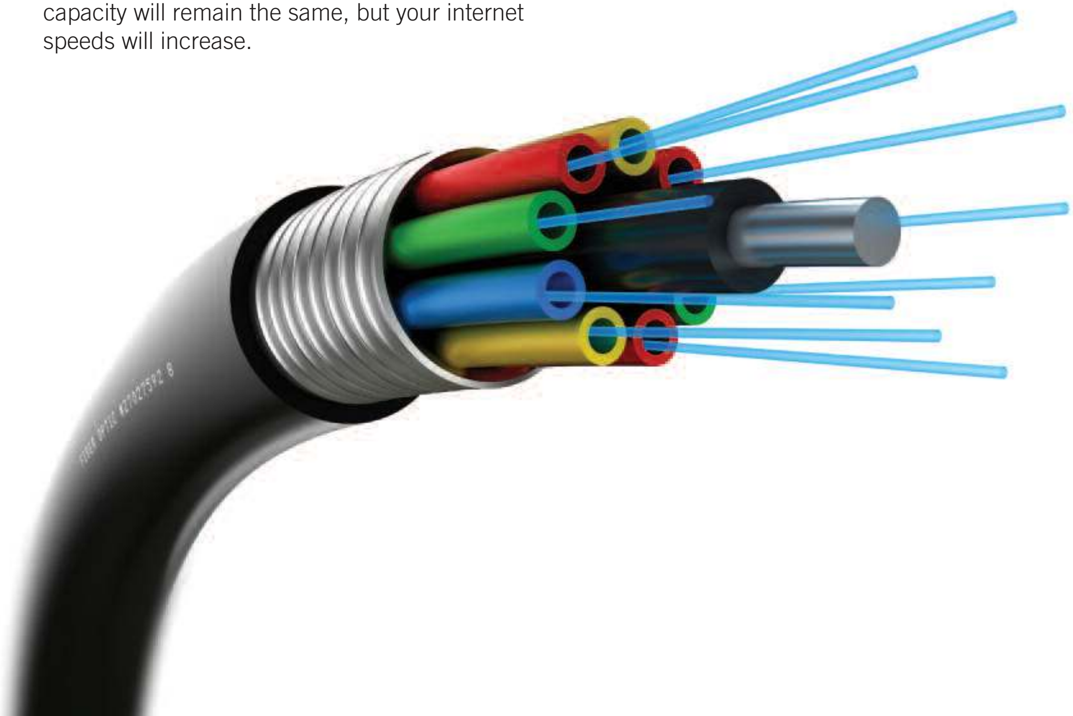
Digital music technology makes heavy use of compression to enable tracks to be stored in ever smaller amounts of memory. Today's ultraportable music players can store hundreds of tracks. Using information about how our brain processes sounds, compression techniques can take a music track and omit the sounds we don't notice: the very high frequencies, the very low frequencies, and any soft note which follows just after a loud sound. This relies on decomposition of sound waves using Fourier series and other techniques from mathematical analysis.

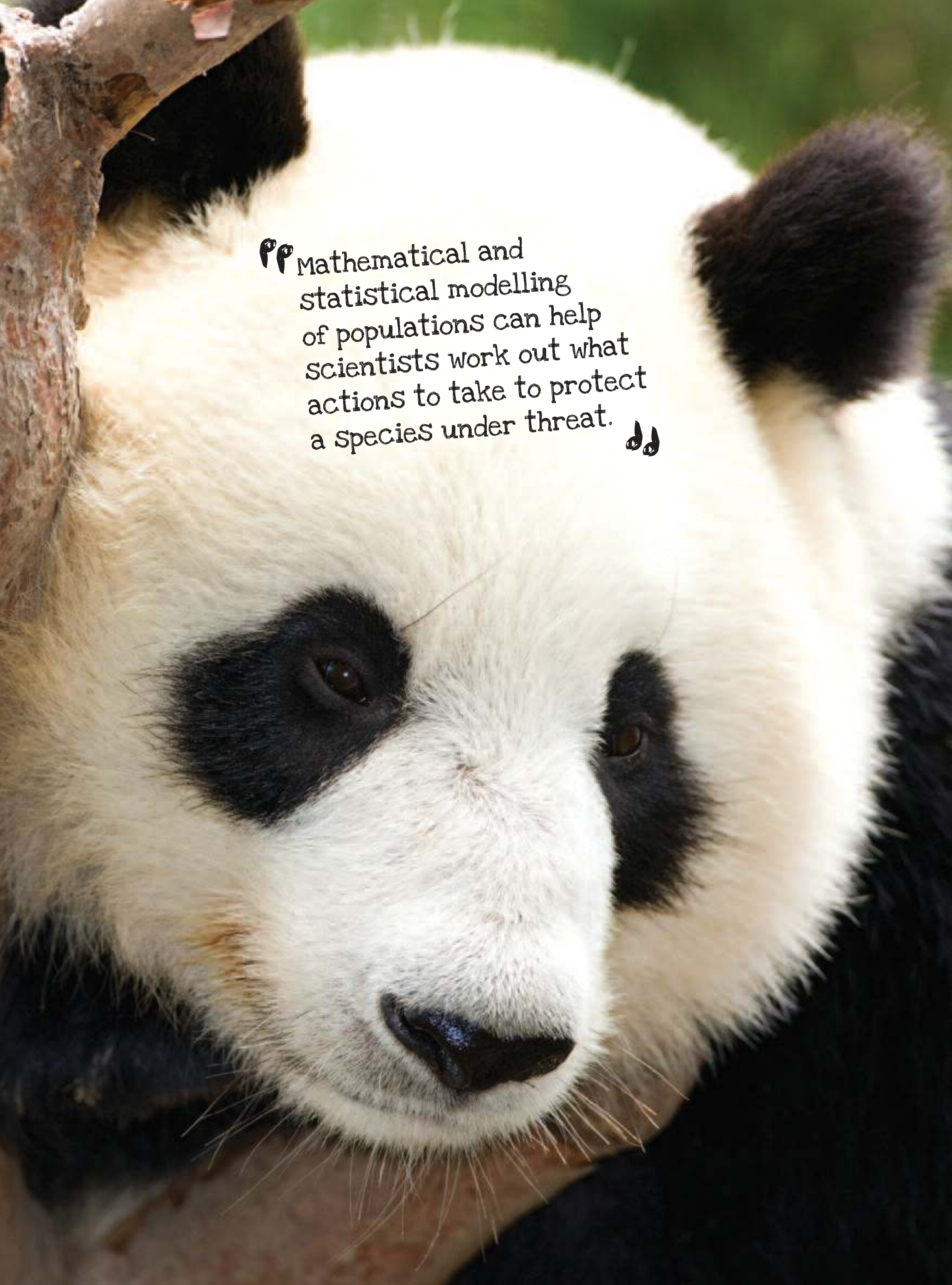


Connection speeds and processing power

Mathematical coding theory describes the maximum possible amount of "clean" or uncorrupted data that can be sent on a particular channel considering the levels of noise. A line will have a set capacity which is filled by data and noise. Shannon's theorem determines the maximum possible internet speeds a computer can receive. It shows that if your internet provider can reduce the amount of noise on your line – for example by using fibre-optic rather than copper cables – the capacity will remain the same, but your internet speeds will increase.

A prediction known as Moore's Law stated in 1965 that the number of transistors on a microchip will double every two years. This is clearly an approximation to a vast and complex industry, but has been a remarkably accurate prediction. The number of transistors is proportional to the processing power and therefore illustrates the exponential growth of processing power. Since internet security and communications technology rely heavily on mathematics, it seems reasonable to assume that job opportunities for mathematicians in this sector will also continue to grow!





Energy and the Environment

The UK target for 2020 is that 15% of its energy should come from renewable sources. Renewable energy sources include solar power, wind and wave power, geothermal energy and hydroelectricity. Government websites suggest that achieving this target could provide £100 billion of investment opportunities and up to half a million jobs in the renewable energy sector by 2020.

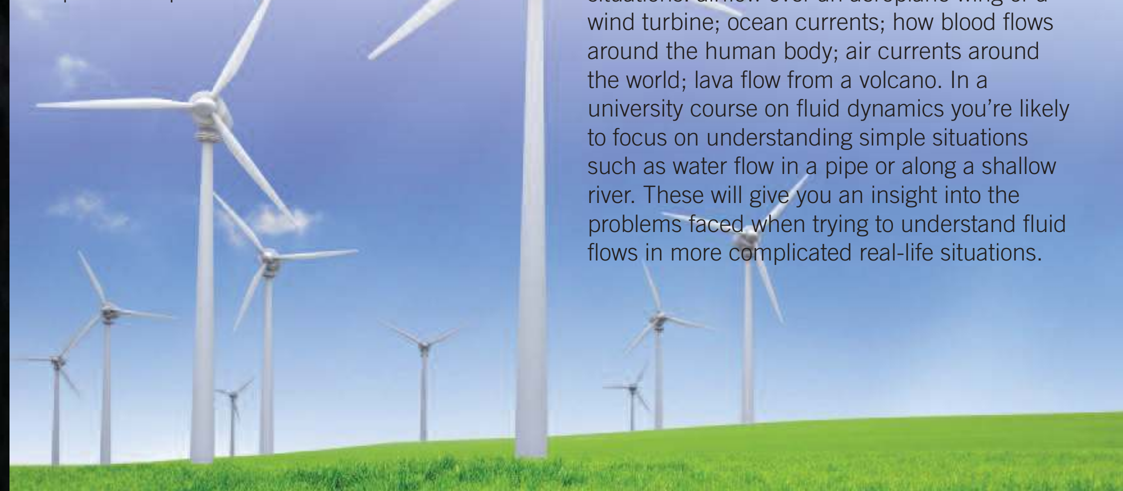
To produce wave power on a large scale it is vital for us to understand how large volumes of water behave. Mathematical theories of fluid dynamics can provide good approximations which are helping engineers design new kinds of power stations far out to sea.

At the same time, many of the world's animal and plant species are under threat from predation or loss of habitat. Mathematical and statistical modelling of populations can help scientists work out what actions to take to protect a species under threat.

Wave and wind power

Mathematicians have studied fluid dynamics for centuries. For example, in 1738, Daniel Bernoulli showed how to relate the pressure, density and speed of a liquid or gas when it is flowing – under certain technical conditions. One of the biggest problems in mathematics is to work out what happens in all the many cases when those technical conditions don't hold. The equations which describe how fluids behave in any situation are known as the Navier-Stokes equations. But there's no proof yet that these equations always have a solution in any given real-world situation. The Clay Mathematics Institute in the USA has called this one of the seven most important open questions in mathematics. It has offered a million-dollar prize for an answer!

The Navier-Stokes equations can be used to create computational models of many different situations: airflow over an aeroplane wing or a wind turbine; ocean currents; how blood flows around the human body; air currents around the world; lava flow from a volcano. In a university course on fluid dynamics you're likely to focus on understanding simple situations such as water flow in a pipe or along a shallow river. These will give you an insight into the problems faced when trying to understand fluid flows in more complicated real-life situations.



Energy and the Environment cont.

Extreme weather events

Conditions in the ocean can sometimes produce freak waves which are a danger to shipping. There are documented examples of waves over twenty-five metres high. That's about the height of a ten-storey building! Such waves can occur without warning and in clear weather. It's not known exactly what causes these waves, although several theories are being developed. In order to develop wave power on a commercial scale, generating turbines out at sea will have to be able to cope with this uncertainty. However, advances in mathematics and computing power can help us design better systems for generating renewable energy.

Tsunamis are generated by events such as underwater earthquakes or landslips which rapidly displace a large volume of water. Coastal areas are in danger when these waves reach shallow water near shore. Some parts of the world are also at high risk of flooding. Monitoring the height of the water at the Thames Barrier allows London to adequately plan further flood defences in the long term, and in the short term to protect London from flooding by raising the Thames barrier.

Seasonal variability

A time series is a collection of observations made sequentially, usually in time. Often in time series, there is some seasonality. For example, the temperature in winter is generally less than that in summer. When assessing climate change, it is important to assess how much of a rise is due to a seasonal effect, and how much is the long term trend we wish to measure.

Seasonal variability also affects other aspects of life. In the UK there are many more jobs in summer available due to agriculture and tourism. In order to assess how unemployment has changed, the Office for National Statistics presents seasonally adjusted figures.

> "Advances in mathematics and computing power can help us design better systems for generating renewable energy"

Order and chaos

The idea of chaos has major applications in a wide range of scientific disciplines. In physics, chaotic behavior can be seen in various small systems such as electrical circuits, lasers and fluid dynamics and in larger systems such as in the dynamics of satellites in the solar system. Changes in the weather also exhibit chaotic behavior, which is why weather predictions may turn out to be inaccurate, as these are based on initial conditions.

A tidal bore occurs when the incoming tide forms a wave that travels up a river or other narrow inlet against the current. The Severn Bore is the most famous example of this phenomenon in the United Kingdom. It can result in a wave which is fifteen or sixteen metres high! However it's quite predictable and follows a nine-year cycle where its timing progresses with respect to the phase of the moon.

The Intermediate Value Theorem is a theorem in calculus or analysis. It's simple but has a variety of applications. You can use it to prove that there are many pairs of antipodal points on the planet that share the exact same temperature at a given point in time. Other theorems in analysis and topology are used to prove important theoretical results in physics and economics. Less seriously, mathematicians have used the Intermediate Value Theorem to investigate when a table with four legs of equal length can be rotated on a smooth but uneven floor so that all four legs rest on the ground.

Modelling the natural world

Mathematical models can help engineers decide how high a flood barrier needs to be, based on statistical models of the risks of floods of varying heights. Modelling can be carried out for many different types of event. Linear regression estimates of earthquake magnitude from data on historical magnitude and length of surface rupture can be obtained by designing a correct regression model.

Doppler radar is used in meteorology. It can detect both precipitation and its motion whether it is a simple rainstorm or a violent tornado. Analysis of wildlife sightings can help predict whether a species is at risk of becoming endangered. Agricultural tests require well-designed experiments. This is an area where techniques from astrophysics, statistics and applied mathematics all provide useful tools.

To infinity and beyond

Mathematics doesn't stop with the techniques, theories and tools taught in any undergraduate programme. Some parts of the subject are thousands of years old and others are brand new! It is impossible for any person now living to understand the proof of every mathematical theorem which has been discovered.

Mathematicians want their results to be beautiful as well as useful: the goal is to find a short proof which really explains why something is true, and which then enables other results to be proved, or a technology to be developed which helps us understand our world or make it a better place.

The Mathematics Subject Classification developed by the American Mathematical Society has thousands of separate topics. Some examples are: 12H20 Abstract differential equations; 52B20 Lattice polytopes; 85A20 Planetary and stellar atmospheres; 94A50 Theory of questionnaires.

"Some parts of the subject are thousands of years old and others are brand new!"

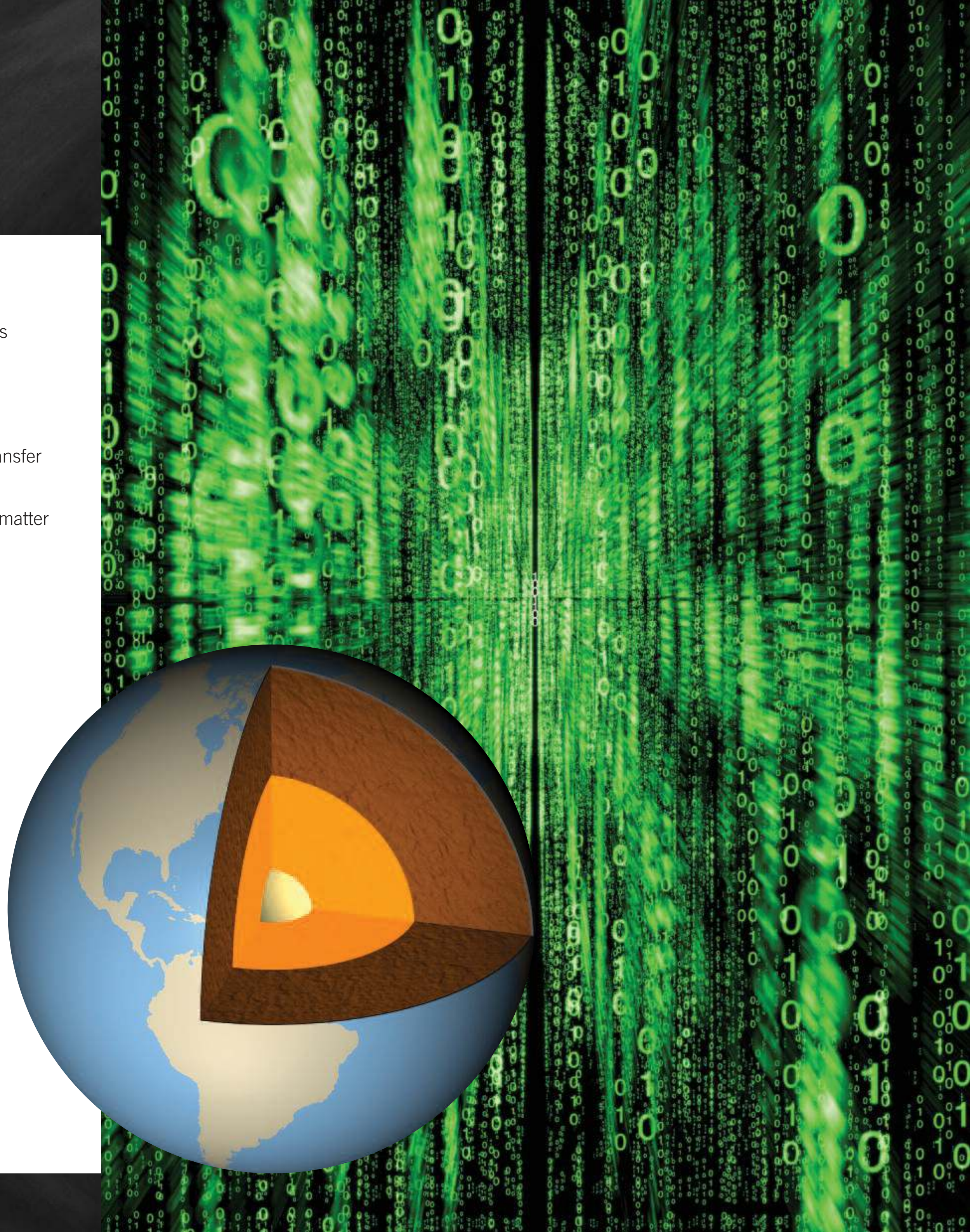
The full list of top-level subject areas is currently as follows. Truly to infinity and beyond!

- 00: General material including elementary mathematics
- 01: History and biography
- 03: Mathematical logic and foundations
- 05: Combinatorics and graph theory
- 06: Order, lattices, ordered algebraic structures
- 08: General algebraic systems
- 11: Number theory
- 12: Field theory and polynomials
- 13: Commutative rings and algebras
- 14: Algebraic geometry
- 15: Linear and multilinear algebra; matrix theory
- 16: Associative rings and algebras
- 17: Nonassociative rings and algebras
- 18: Category theory, homological algebra
- 19: K-theory
- 20: Group theory and generalizations
- 22: Topological groups, Lie groups
- 26: Real functions and elementary calculus
- 28: Measure and integration
- 30: Functions of a complex variable
- 31: Potential theory
- 32: Several complex variables and analytic spaces

To infinity and beyond cont.

- 33: Special functions including trigonometric functions
- 34: Ordinary differential equations
- 35: Partial differential equations
- 37: Dynamical systems and ergodic theory
- 39: Difference and functional equations
- 40: Sequences, series, summability
- 41: Approximations and expansions
- 42: Fourier analysis
- 43: Abstract harmonic analysis
- 44: Integral transforms, operational calculus
- 45: Integral equations
- 46: Functional analysis
- 47: Operator theory
- 49: Calculus of variations and optimal control; optimization
- 51: Geometry, including classic Euclidean geometry
- 52: Convex and discrete geometry
- 53: Differential geometry
- 54: General topology
- 55: Algebraic topology
- 57: Manifolds and cell complexes
- 58: Global analysis, analysis on manifolds
- 60: Probability theory and stochastic processes
- 62: Statistics
- 65: Numerical analysis
- 68: Computer science
- 70: Mechanics of particles and systems
- 74: Mechanics of deformable solids
- 76: Fluid mechanics
- 78: Optics, electromagnetic theory
- 80: Classical thermodynamics, heat transfer
- 81: Quantum Theory
- 82: Statistical mechanics, structure of matter
- 83: Relativity and gravitational theory
- 85: Astronomy and astrophysics
- 86: Geophysics
- 90: Operations research, mathematical programming
- 91: Game theory, economics, social and behavioral sciences
- 92: Biology and other natural sciences
- 93: Systems theory; control
- 94: Information and communication, circuits
- 97: Mathematics education

See **www.maths.qmul.ac.uk/undergraduate/impact** for full details of references used in producing this book.



This publication has been printed on environmentally friendly material from sustainable sources

This publication has been produced by the Publications and Web Office for the School of Mathematical Sciences

For further information contact:
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London
E1 4NS
Tel: +44 (0)20 7882 5440
Fax: +44 (0)20 7882 7684
email: curriculum-impact@maths.qmul.ac.uk
www.maths.qmul.ac.uk

Queen Mary
University of London