Quadratic Equations in Sport

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Starter worksheet

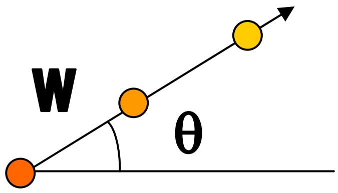
Lots of sports involve kicking, hitting or throwing a ball so it hits a target: basketball, football, golf, netball, rugby, shotputting, squash, table tennis, tennis, volleyball… and more!

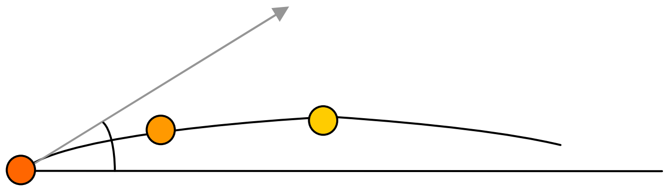
What all of these sports have in common is that practice makes perfect: to get the ball to hit the target you have to throw or kick it at the right speed and angle. But thinking about the maths involved can be useful! It can help you understand whether you need to be more precise with the angle or whether you need to work out in the gym so you can hit the ball at a higher speed.

Suppose you throw a ball with speed **w** and at an angle **θ**. Just think about how the height of the ball varies. If the angle is close to zero, the ball will go up a little bit and then start to fall to the ground. If the angle is close to 90 degrees, the ball will go up a lot but it will land close to where you started.

It turns out that if you want to get the ball as far as possible before it lands, you need to throw it with an angle close to 45 degrees, and as hard as you can.

The ball follows a path which can be written down as a quadratic equation. At the start the ball travels at angle **θ**, but then gravity begins to pull it down and it travels at shallower (and smaller) angles. We're assuming that there isn't anything else affecting the ball, so for example it isn't being blown around by a strong wind.





Let's think about quadratic equations and what they look like.

Quadratic equations have the form $\mathbf{y = ax^2 + bx + c}$ where **x** and **y** are variables and **a**, **b** and **c** are coefficients. For any particular equation this means that we can imagine that **a**, **b** and **c** are fixed numbers. The graph of the curve shows how the value of **y** changes if we change **x**.

If **a** is a negative number then the graph of the curve will go up to a maximum point and then down.
If **a** is a positive number then the graph of the curve will go down to a minimum point and then up.

1. What happens if $\mathbf{a = 0}$?

2. Think about the curve $\mathbf{y = 10 - x^2}$ for **x** between the values $\mathbf{x = -5}$ and $\mathbf{x = 5}$.
   (a) In this equation $\mathbf{b = 0}$. What are the values of **a** and **c**?
   (b) What is the maximum possible value of **y** satisfying the equation?
   (c) What is the minimum possible value of **y** for **x** in the range given?
   (d) Sketch the curve for **x** in the range given.

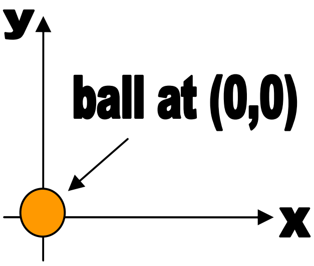
3. Suppose that we want an equation that passes through the point where $\mathbf{x=0}$ and $\mathbf{y=0}$.
   What value of **c** in the equation $\mathbf{y = ax^2 + bx + c}$ do we need for this to happen?

Quadratic Equations in Sport

Core worksheet

If you hit a ball into the air – like kicking a football towards the goal or a tennis ball back towards your opponent - and want to calculate exactly where it will land, you need a quadratic equation. In real life you wouldn't be out there with a calculator, but if you were programming a sports game you'd have to give the computer some equations so it could figure out how to make the gameplay feel real.

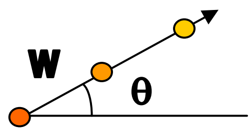
We can use a two-dimensional coordinate system with **x** measuring the horizontal distance along the ground and **y** the vertical distance up into the air.



Suppose that the point (0,0) – where **x=0** and **y=0** - is the point where we hit or kick the ball. The equation $\mathbf{y = ax^2 + bx}$ will always pass through the point **x=0**, **y=0**.

If we let **x** be the horizontal distance the ball travels, and **y** be the vertical distance, then **x** and **y** are related by a quadratic equation. The equation depends on a number **g** which measures the pull of gravity, on **w** which is the speed you hit or kick the ball at, and on **θ** which is the angle you hit or kick it at.

$$y = \frac{-g}{2w^2(\cos\theta)^2}x^2 + (\tan\theta)\,x$$



The equation is given above. It may look a bit complicated but it's really just $\mathbf{y = ax^2 + bx}$ with **a** and **b** chosen to match the situation. Let's work through some examples. Suppose that **x** and **y** are measured in metres, the speed **w** in metres/second, and that gravity **g** is 9.8 m/s^2. Assume we kick or hit the ball from ground level.

1. The vertical distance is zero at the start, when **x** = 0 and **y** = 0. It's also zero when the ball hits the ground again. In terms of **w**, **θ** and **g**, what is **x** when **y** = 0 for the second time?

   (Hint: This is just like solving 0 = $\mathbf{ax^2 + bx}$ for **x** in terms of **a** and **b**.)

2. Suppose that **g** = 9.8 m/s^2 and that **w** = 10 m/s^2. By subsituting these values into the equation above, write down the equation for the path of the ball when
   - (a) **θ** = 5 degrees
   - (b) **θ** = 30 degrees
   - (c) **θ** = 45 degrees
   - (d) **θ** = 60 degrees
   - (e) **θ** = 85 degrees

3. For each of the angles in Question 2, how far does the ball travel horizontally before it reaches the ground?

4. Of the five angles tested, which allows the ball to be thrown or kicked the furthest distance?

5. How else could you get the ball to go further?